Midterm II Review Session

CS 171

March 15, 2024

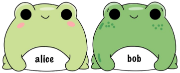


Table of Contents

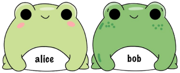
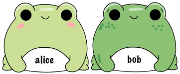


Table of Contents

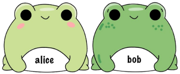


MAC: The Concept

So far in the class, we've precisely defined confidentiality for end-to-end encrypted messaging with *symmetric-key encryption*.

But how can we guarantee the **integrity** of a ciphertext?

A Message Authentication Codes (MAC) is a keyed checksum, which is sent along with the message. It takes in a fixed-length secret key and an arbitrary-length message, and outputs a fixed-length checksum. A secure MAC has the property that any change to the message will render the checksum invalid.



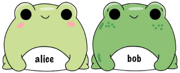
MAC: Definition

A MAC scheme consists of 3 PPT algorithms (Gen, MAC, Verify):

- $Gen(1^n)$: Outputs a key k.
- $MAC_k(m)$: Outputs a tag t.
- $Verify_k(m, t)$: Outputs 0/1.

These satisfy 2 properties:

1. **Correctness:** $\forall n, k \leftarrow Gen(1^n), \forall m \in \{0,1\}$, we have that $Verify_k(m, MAC_k(m)) = 1$.
2. **Security:** $Verify_k(m, t)$ outputs 1 if and only if $MAC_k(m) = t$.

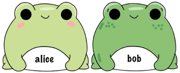


MAC: Security Game

The adversary's goal is to **forge** a MAC. The adversary wins only if they output a valid tag on a message that was never previously queried.

The game is between a challenger C and the adversary A.
$\text{MACForge}_{A;\Pi}(1^n)$:

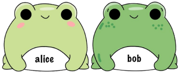
1. C samples $k \quad Gen(1^n)$.
2. A makes MAC queries to the challenger. Let M be the list of queries A makes.
3. Finally, A outputs $(m \; ; t \;)$.
4. C outputs 1 if $Verify(m \; ; t \;) = 1 \wedge m \; \notin M$ and 0 otherwise.



MAC: Security Definition

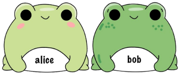
$\Pi = (Gen; MAC; Verify)$ is existentially unforgeable under the adaptive chosen attack if *8* PPT A it holds that:

$$\Pr[\mathrm{MACForge}_{A;\Pi} = 1] \quad \mathrm{negl}(n)$$



MAC: Tips

1. If you are asked to construct a new MAC' and prove its security:
   - Use the system from the proof workshop where your secure underlying building block is the MAC.
   - Assume there is an adversary A that breaks MAC'.
   - Construct an external adversary B that simulates the MACForge game for A and uses this to break MAC. Contradiction!
   - **Hint:** B can *tinker* with the what it gets from A and what it forwards from its oracle to A.
2. There can be interesting **variations** of unforgeability such as strong unforgeability from Discussion 6, Q2: Adversary can win even if they output a valid tag on a message that was previously queried.
3. You can be asked to **compare** the security properties of the MAC security definition with a new primitive.
   - E.g. define a primitive x that is not a MAC.

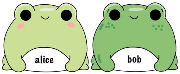


MAC: Practice Problem (Part (a))

Spring 2021 MT2 Q2

Consider a "CCA-style" extension to the definition of secure message authentication codes, where the adversary is provided with both a *MAC* and a *Verify* oracle. Our starting point will be the "standard" notion of MAC security, called "existential unforgeability under adaptive chosen-message attacks," and we will consider a variant of this definition that allows for Verify oracle queries.

(a) Provide a formal definition of CCA-secure MACs. That is, describe an experiment called CCA Mac Forge$_{A;\Pi}(n)$, and provide a security requirement stating that no adversary can win your game except with negligible probability.



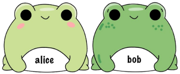
MAC: Practice Problem (Part (a) Solution)

(a) Provide a formal definition of CCA-secure MACs. That is, describe an experiment called $\text{CCA} \quad \text{Mac} \quad \text{Forge}_{A;\Pi}(n)$, and provide a security requirement stating that no adversary can win your game except with negligible probability.

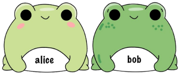
1. The challenger samples $k \quad \text{Gen}(1^n)$.
2. The adversary A is given input 1^n and oracle access to $\text{Mac}_k(\ )$ and $\text{Verify}_k(\ ;\ )$. The adversary eventually outputs a pair $(m; t)$. Let Q denote the set of all queries that A asked to its $\text{Mac}_k(\ )$ oracle.
3. The output of the experiment is defined to be 1 if and only if (1) $\text{Verify}_k(m; t) = 1$ and (2) $m \notin Q$.

Π is a CCA-secure MAC if for all adversaries A,

$$\Pr[\text{CCA} \quad \text{Mac} \quad \text{Forge}_{A;\Pi}(n) = 1] = \text{negl}(n).$$



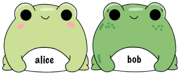
(b) Assume that Π is a standard secure *deterministic* MAC that has *canonical verification*, meaning that i) the Mac algorithm is deterministic and ii) the Verify algorithm, on input $(m; t)$, recomputes $t' := \mathsf{Mac}_k(m)$ and accepts if $t' = t$. Prove that Π also satisfies your definition from part (a).



MAC: Practice Problem (Part (b) Solution)

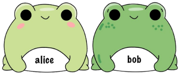
When Π is deterministic and has canonical verification, each message has only a single valid tag. Thus, if the scheme is secure, then access to a Verify oracle does not help (and so Π is secure in the sense of the definition given in part (a)). To see this, note that for any query $(m; t)$ to the Verify oracle there are 3 possibilities:

1. m was previously queried to the Mac oracle, and response t was received. Here the adversary already knows that $\text{Verify}_k(m; t) = 1$.
2. m was previously queried to the Mac oracle, and response $t' \neq t$ was received. Since Π is deterministic, the adversary already knows $\text{Verify}_k(m; t) = 0$.
3. m was not previously queried to the Mac oracle. By security of Π, we can argue that $\text{Verify}_k(m; t) = 0$ with all but negligible probability because otherwise, $m; t$ is a valid forgery. Let's prove it.

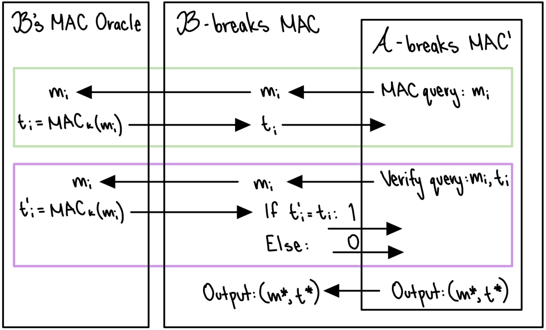


MAC: Practice Problem (Part (b) Solution Continued)

We want to show that if m was not previously queried to the Mac oracle, by security of Π, we can argue that $\text{Verify}_k(m; t) = 0$ with all but negligible probability because otherwise, $m; t$ is a valid forgery.
Let MAC^0 be a CCA-secure MAC. Assume that $\text{Verify}_k(m; t) = 1$. Then there exists an adversary A that can query a message m to the verify oracle in the CCA-secure MAC scheme to obtain a valid MAC.
Now construct an adversary B that simulates the security game for A to win the Π security game.



MAC: Practice Problem (Part (b) Solution Continued)



We successfully simulate the game for A because its queries are accurately answered. So A can produce a message that was not previously queried such that $\text{Verify}_k(m; t) = 0$, then so can B. Contradiction.

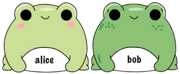
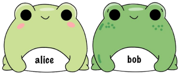


Table of Contents

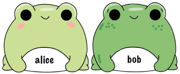


CRHF: Basic Definitions

- Syntax:

$$H^s(x) = y$$

- A **collision** in H^s is a pair (x, x') such that $x \neq x'$ but $H^s(x) = H^s(x')$.
- H is guaranteed to have collisions. We require that $|y| < |x|$ (H is **compressing**).
- If it's hard to find those collisions, then the hash function is **collision-resistant**.

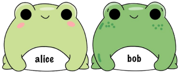


CRHF: Formal Syntax

- The hash function H is a pair of algorithms: $H = (\mathrm{Gen}; H)$.
- Gen: outputs a random key/seed s:

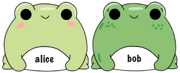
$$s \qquad \mathrm{Gen}(1^n)$$

  The key is allowed to be public.
- H^s: This is also sometimes referred to as the hash function.
  The output length – and sometimes the input length – are fixed.
  H^s is deterministic.



CRHF: Security Game

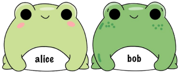
- **Summary:** The adversary is given s and a description of H, and they try to find a collision in H^s with non-negligible probability.
- Hash-coll$_{A,H}(n)$:
  1. The challenger samples a key $s \leftarrow \mathsf{Gen}(1^n)$ and gives s to the adversary A.
  2. A produces two inputs (x, x') to H^s.
  3. A wins (and the game outputs 1) if (x, x') are a collision:
  $$x \neq x' \text{ and } H^s(x) = H^s(x')$$
  Otherwise, A loses (the game outputs 0).
- Note that the adversary can compute H^s by themselves.



CRHF: Security Definition

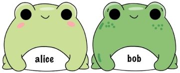
- H is **collision-resistant** if for any PPT adversary A, there is a negligible function negl such that:

$$\Pr[\text{Hash-coll}_{A;H}(n) = 1] \quad \text{negl}(n)$$



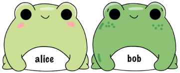
CRHF: Tips

- The adversary in the CRHF security game is given s and a description of H, so they can compute $H^s(x)$ on any input x of their choosing.



CRHF: Practice Problem

- Summary: The problem shows you how to reprogram a hash function so that a given x maps to a given y, while maintaining collision-resistance.
- Source: Midterm 2, Fall 2019, Q 5.2.b

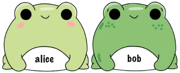


The problem:

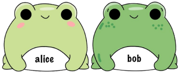
- Let $H = (\mathsf{Gen}, H)$ be a CRHF. Let x belong to the domain of H^s, and let y belong to the range of H^s.
- Next, for any s $\mathsf{Gen}(1^n)$:

$$\text{let } H_1^s(x) = \begin{cases} y & \text{if } x = x \\ H^s(x\;) & \text{if } x \neq x \text{ and } H^s(x) = y \\ H^s(x) & \text{otherwise} \end{cases}$$

- Prove that (Gen, H_1) is a CRHF.



CRHF: Practice Problem



CRHF: Practice Problem Solution

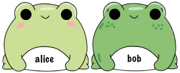
Theorem

$(\mathrm{Gen}; H_1)$ *is a CRHF.*

Proof:

<u>Overview:</u>

- Assume toward contradiction that $(\mathrm{Gen}; H_1)$ is not a CRHF. Then there exists an adversary A that wins the CRHF game for H_1 (by finding a collision in H_1) with non-negligible probability.
- We will use A to construct an adversary B that wins the CRHF game for H with non-negligible probability.
- This is a contradiction because $(\mathrm{Gen}; H)$ is a CRHF. So our initial assumption was false and $(\mathrm{Gen}; H_1)$ is also a CRHF.



CRHF: Practice Problem Solution

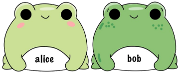
<u>Construction of B:</u>

1. In the CRHF game for H, the challenger samples $s \leftarrow \mathsf{Gen}(1^n)$ and gives s to the adversary B.
2. B will run A on input s until A produces two inputs (x, x').
3. B makes a list of collision candidates:

$$C := \{(x, x'), (x, x\ \ ), (x', x\ \ )\}$$

   and checks whether each candidate $(x_1, x_2) \in C$ satisfies the conditions: $x_1 \neq x_2$ and $H^s(x_1) = H^s(x_2)$.

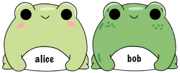
4. B outputs the first candidate $(x_1, x_2) \in C$ that satisfies the conditions.



- Note that with non-negligible probability (x, x') will be a collision in H_1^s:

$$x \neq x' \text{ and } H_1^s(x) = H_1^s(x')$$

- We will prove that in this case, B will succeed in finding a collision in H^s.



CRHF: Practice Problem Solution

CRHF: Practice Problem Solution

Let's assume that (x, x') are a collision in H_1^s. Then consider the following trivial cases:

- Case 1: $H^s(x) = y$: In this case, $H_1^s = H^s$; reprogramming the function doesn't do anything. If (x, x') are a collision in H_1^s, then (x, x') will be a collision in H^s. For the remaining cases, assume that $H^s(x) \neq y$.
- Case 2: $x = x$ or $x' = x$: This will not happen if (x, x') is a collision in H_1^s because x is the only input that H_1^s maps to y.

CRHF: Practice Problem Solution

Now consider some more-interesting cases:

Case 3: $(x, x') \in A$. Then

$$H^s(x) = y = H^s(x')$$

so (x, x') are a collision in H^s.

Case 2: $(x, x') \in B \cup C$. Then

$$H^s(x) = H_1^s(x) = H_1^s(x') = H^s(x')$$

so (x, x') are a collision in H^s.

CRHF: Practice Problem Solution

Case 4: $x \in A, x' \in B$. Then

$$H^s(x') = H^s(x\ )$$

so $(x', x\ )$ are a collision in H^s.

Case 5: $x \in B, x' \in A$. Then

$$H^s(x) = H^s(x\ )$$

so $(x, x\ )$ are a collision in H^s.

Table of Contents

OWF: Definition

Syntax:

$$f(x) = y$$

A function $f : \{0,1\}^* \to \{0,1\}^*$ is *one-way* if

- It's easy to compute, i.e., computing $f(x)$ runs in “probabilistic polynomial time.”, but
- It's hard to invert, i.e., there is no “probabilistic polynomial time” algorithm that can compute $f^{-1}(y)$.
- Note: $\{0,1\}^* \to \{0,1\}^*$ means the input and output can be arbitrarily long bit strings.

OWF: Security

- How can we formally define “hard to invert”?
- $\underline{\text{OWF-Sec}_{\mathcal{A},f}(n)}$:
  - The challenger randomly samples an input $x \leftarrow \{0,1\}^n$ and gives $f(x)$ to the adversary $\mathcal{A}$ along with 1^n.
  - $\mathcal{A}$ produces a value $x' \in \{0,1\}^n$.
  - $\mathcal{A}$ wins (and the game outputs 1) if $f(x') = f(x)$
  - Otherwise, $\mathcal{A}$ loses (the game outputs 0).
- The probability $\mathcal{A}$ wins the above game should be at most $\mathsf{negl}(n)$ for f to be secure.
- This can be expressed equivalently as:

$$\Pr_{x \leftarrow \{0,1\}^n}\left[\mathcal{A}(1^n, f(x)) \in f^{-1}(f(x))\right] \leq \mathsf{negl}(n).$$

OWF: Tips

OWF's are \almost universal" in the sense that most cryptographic primitives imply the existence of OWFs.

If a question asks you to construct a OWF from a standard-looking primitive, you probably do it.

The only gotcha is if the given primitive is contrived, e.g. constructing a OWF f from a PRPF F as follows:

$$f(x_0 \| x_1) = F(x_0; x_1)$$

See discussion 8 for detail on why this example fails.

OWF: Example Questions

Example questions: construct a one-way function from one of the following primitives:

- A PRG $G : \{0,1\}^{n/2} \to \{0,1\}^n$
- a CRHF (Gen, H) where $H^s : \{0,1\}^n \to \{0,1\}^{n/2}$
- a one-to-one function (permutation) $F : \{0,1\}^n \to \{0,1\}^n$ with a hard-concentrate predicate $hc(\cdot)$.

OWF: Practice Problem

Question: construct a one-way function from a CRHF (Gen, H) such that $H^s : \{0,1\}^n \to \{0,1\}^{n/2}$.

We'll prove the following theorem:

[redacted]

$f(s \| x) = s \| H^s(x)$ is a OWF.

Proving $f(s \| x) = s \| H^s(x)$ is a OWF

Step 1: Stating our argument.

- Suppose for the sake of contradiction that f is not a OWF.
- This implies that there exists an adversary $\mathcal{A}$ that can win the OWF-Sec$_{\mathcal{A},f}(n)$ security game with nonnegl(n) probability.
- We will construct an adversary $\mathcal{B}$ from $\mathcal{A}$ that wins Hash-coll$_{\mathcal{A},H}(n)$ with nonnegl(n) probability.

Proving $f(s \| x) = s \| H^s(x)$ is a OWF

Step 2: Construction of B:

- B is given the truly random seed s from the CRHF challenger.
- B samples a random $x \leftarrow \{0,1\}^n$ and runs A on $H^s(x)$ to obtain x'.
- If $x = x'$, abort.
- Otherwise, output (x, x') as a collision.

Proving $f(s \, k \, x) = s \, k \, H^s(x)$ is a OWF

Step 3: Analysing B:

- We need to lower bound the probability that we don't abort (i.e., the probability we win).
- First, observe that the probability our random x collides with x^0 by chance ($H^s(x) = H^s(x^0)$) is upper bounded by the birthday bound, $2^{\ n=2}$. Note: we have no control over the particular x^0 that A got from inverting $f(x)$, but the x that B sampled itself is uniformly random, meaning that chance $x = x^0$ is still random even if A doesn't choose x^0 randomly.
- Conditioned on the above not happening, the probability that $x \; 6 \; x^0$ is at least $1{=}2$. This follows from the fact that H takes n bits to $n{=}2$ bits, implying $\Pr[x = x^0] = \frac{1}{2^{n=2}} < \frac{1}{2}$.
- Putting these two point together:

$$\Pr[x \; 6 \; x^0 | H^s(x) = H^s(x^0)] \quad \frac{1}{2} \quad \frac{1}{2^{n=2}} :$$

Proving $f(s,k,x) = s,k,H^s(x)$ is a OWF

Step 4: Wrapping up:

- We proved that we don't abort probability $\geq \frac{1}{2} - \frac{1}{2^{n/2}}$.
- In the case that B doesn't abort, it follows from the construction that (x, x') are a valid collision.
- Thus,

$$
\begin{aligned}
& \Pr[\text{Hash-coll}_{B,H}(n) = 1] \\
& \geq \Pr[\text{OWF-Sec}_{A,f}(n) = 1] - \Pr[x \neq x' \mid H^s(x) = H^s(x')] \\
& = \text{nonnegl}(n) \cdot \frac{1}{2} - \frac{1}{2^{n/2}} \\
& = \text{nonnegl}'(n)
\end{aligned}
$$

- In summary, given an adversary A that wins $\text{OWF-Sec}_{A,f}(n)$ with non-negligible probability, B wins $\text{Hash-coll}_{B,H}(n)$ with non-negligible probability, which is a contradiction.

Table of Contents

Public Key Encryption: Definition

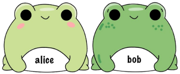
(The syntax and most properties are very similar to private/symmetric key encryption that we've seen earlier.)

A PKE scheme consists of three PPT algorithms $(Gen; Enc; Dec)$ where

- $Gen(1^n)\ \ !\ \ (\mathbf{sk}; \mathbf{pk})$
- $Enc(\mathbf{pk}; m)\ \ !\ \ c$
- $Dec(\mathbf{sk}; c)\ \ !\ \ m = ?$

and these satisfy two properties

- **Correctness**: $Dec(sk; Enc(pk; m)) = m$.
- **Security**: EAV = CPA security / CCA security



PKE: CPA Security

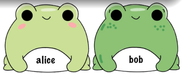
- Challenger samples $(sk; pk)$ $Gen(1^n)$ and gives pk to A.
- A outputs two messages $m_0; m_1$.
- Challenger samples a bit b 2 $f0; 1g$ and outputs $Enc(pk; m_b)$.
- A outputs b^0 as a guess for b.

CPA-secure if for all PPT A

$$Pr[b^0 = b] \quad \frac{1}{2} + negl(n)$$

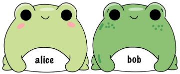
Intuition

Looking at the ciphertext should not reveal which message was encrypted.



Tips

- pk is given to the adversary, so no encryption oracle is needed – A can locally encrypt whatever it wants.
- sk is unknown, so decryption is not possible – CCA game for PKE gives access to a decryption oracle to A.
- Most proof techniques are similar to that of private key encryption schemes:
  - Show that a certain scheme is not CPA/CCA secure – construct an adversary for the game that is able to figure out which message was encrypted.
  - Show that a certain scheme is secure – often relies on the security of some other primitive ! *Proof by contradiction*.



PKE Example: El Gamal Encryption

PKE scheme based on DDH.

- $Gen(1^n)$: Generate cyclic group $\mathbb{G}$ of order q and a generator g.
  Sample $x \; 2 \; \mathbb{Z}_q$ and $h = g^x$.
  Output $pk = (\mathbb{G}; q; g; h)$; $sk = x$
- $Enc(pk; m) \; ! \; (c_1; c_2)$: Sample $r \; 2 \; \mathbb{Z}_q$.
  Output $(c_1; c_2) = (g^r; m \;\; h^r)$
- $Dec(sk; (c_1; c_2)) \; ! \; m$: **Output** $m = \frac{c_2}{c_1^x}$

Correctness

$$Dec(sk; Enc(pk; m)) = Dec(sk; (g^r; mh^r)) = \frac{mh^r}{(g^r)^x} = \frac{mh^r}{h^r} = m$$

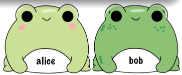
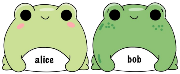


Table of Contents

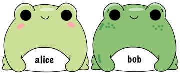


Key Exchange

Consists of three randomized algorithms $(P_1; P_2; P_3)$:

1. Alice computes $(m_1; st) \quad P_1(1^n)$ and sends m_1 to Bob.
2. Bob computes $(m_2; k) \quad P_2(m_1)$. Then he sends m_2 to Alice and outputs k.
3. Alice computes $k \quad P_3(st; m_2)$ and outputs k.

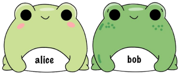
- **Correctness**: Both parties get the same key k.
- **Security**: No eavesdropper can distinguish between $(m_1; m_2; k)$ and $(m_1; m_2; r)$ where r is a random element.



Problem: Key exchange from CPA-Secure PKE

Question

Given a PKE scheme $(Gen; Enc; Dec)$, construct a secure key exchange scheme $(P_1; P_2; P_3)$.



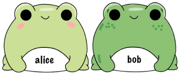
Problem: Key exchange from CPA-Secure PKE

Question

Given a PKE scheme $(Gen; Enc; Dec)$, construct a secure key exchange scheme $(P_1; P_2; P_3)$.

Construction

$P_1(1^n)$: Run $Gen(1^n)$! $(sk; pk)$. Return $(m_1; st) = (pk; sk)$.
$P_2(m_1)$: Sample random r and run $Enc(m_1; r)$! c.
    Return $(m_2; k) = (c; r)$.
$P_3(m_2; st)$: Run $Dec(st; m_2)$! r^0 and return r.



Solution: Key exchange from CPA-Secure PKE

By contradiction: Suppose the Key exchange scheme is not secure. Then we have A that can distinguish $(m_1; m_2; k)$ from $(m_1; m_2; r)$ where r is random.
We'll construct B for the CPA game that distinguishes between encryptions of m_0 or m_1.